Categories

We won't be needing much category theory, but it's good to know a few basic definitions.

A <u>category</u> is a collection of <u>objects</u> (e.g. A, B, C) and for each pair of objects a collection of <u>morphisms</u> or <u>arrows</u> $\mathrm{Mor}(A,B)$ such that:

1.) There is composition of morphisms $\mathrm{Mor}(A,B) \times \mathrm{Mor}(B,C) \to \mathrm{Mor}(A,C)$ that is associative: $(f \circ g) \circ h = f \circ (g \circ h)$

2.) Every object A has an identity morphism $id_A \in \mathrm{Mor}(A,A)$ s.t. $f \circ id_A = id_B \circ f = f \quad \forall \ f \in \mathrm{Mor}(A,B)$.

Check: id_A is unique.

Examples

1.) Category of sets, $\mathrm{Mor}(A,B)$ all maps $f: A \to B$

2.) Category of sets w/ $\mathrm{Mor}(A,B) =$ set of relations $f \subset A \times B$.

3.) finite-dim'l vector spaces $/\mathbb{R}$ w/ linear maps

4.) groups w/ group homomorphisms

5.) topological spaces w/ continuous maps

Def: $f \in \mathrm{Mor}(A,B)$ is an <u>isomorphism</u> if $\exists \ g \in \mathrm{Mor}(B,A)$ s.t. $g \circ f = id_A$ and $f \circ g = id_B$.

The set of isomorphisms from A to itself is the <u>automorphism group</u> of A, Aut(A).

Check: inverse of an isom. is unique, and Aut(A) is a group.

Ex:

1.) A = set w/ n elements, then $\mathrm{Aut}(A) \cong S_n$.

2.) V = dim n vector space over $\mathbb{R}$, then $\mathrm{Aut}(A) \cong GL_n(\mathbb{R})$

Claim: If $A \cong B$, then $\mathrm{Aut}(A) \cong \mathrm{Aut}(B)$ (which depends on the choice of map $A \to B$).

Claim/Exercise: group is a category w/ one object where all morphisms are isomorphisms.
(Think of the arrows as the elements of the group, and multiplication = composition)

Def: A <u>groupoid</u> is a category in which every morphism is an isomorphism.

(e.g. sets w/ bijections, groups w/ isomorphisms, top. spaces w/ homeomorphisms)

The main difference between a group + groupoid is that we can't always "multiply" two elements. i.e. if our operation is $*$, it satisfies the following groupoid axioms:

1.) If $f*(g*h)$ is defined, then so is $(f*g)*h$, and their equal.

2.) $f*f^{-1}$ and $f^{-1}*f$ are always defined.

($\text{if } f: A \to B,\ f*f^{-1} = id_B$ and $f^{-1}*f = id_A$)

3.) $f*id_A = id_B*f = f$.

Functors

In algebraic topology, the idea is to associate to a topological space X a discrete algebraic object $A(X)$ to tell spaces apart.

We want $X \mapsto A(X)$ to be a "functor" so that

if $f: X \to Y$ is continuous, we get an induced morphism

$A(f): A(X) \to A(Y)$ in the category that $A(X)$ lives in.

More precisely...

Def: Let C and D be categories. A functor F from C to D associates to each object X in C an object $F(X)$ in D, and to each morphism $f: X \to Y$ in C a morphism $F(f): F(X) \to F(Y)$ in D s.t.

1.) $F(id_X) = id_{F(X)}$ (preserves identity)

2.) $F(g \circ f) = F(g) \circ F(f)$. (preserves composition)

Basic examples:

1.) The forgetful functor F takes a group to its underlying set.

2.) Tensor product: In the category of vector spaces over a field k, fix V. Then $W \longmapsto W \otimes V$ gives a functor from the category to itself (so does $W \mapsto V \otimes W$)

3.) $W \longmapsto \mathrm{Hom}_k(V, W)$ is also a functor.

If $f: W \to Y$ is linear, then the induced map $\mathrm{Hom}(V,W) \longrightarrow \mathrm{Hom}(V,Y)$

$$\begin{tikzcd} V \arrow[d, "a"'] \arrow[dr] & \\ W \arrow[r, "f"'] & Y \end{tikzcd}$$

is given by $a \longmapsto f \circ a$

4.) Free functor: there's a functor F from sets to groups that give the <u>free</u> <u>group</u> generated by X

e.g. $F(\{a,b\}) = \langle a, b \rangle =$ finite length "words" in a, b, a^{-1}, b^{-1},

where multiplication is concatenation.

i.e. $a * (ab) = a^2 b$, $a^3 * (a^{-4} b a) = a^{-1} b a$

$\left(F(\{a\}) \cong \mathbb{Z}. \right)$